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Sheaves of Einstein Algebras

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To include all types of singularities into a geometrically tractable theoretical scheme we change from Einstein algebras, an algebraic generalization of general relativity, to sheaves of Einstein algebras. The theory of such spaces, called Einstein structured spaces, is developed. Both quasiregular and curvature singularities are studied in some detail. Examples of the closed Friedmann world model and the Schwarzschild spacetime show that Schmidt's b-boundary is a useful theoretical tool when considered in the category of structured spaces.

INTRODUCTION

In previous work (Heller, 1992) we defined an *Einstein algebra* $\mathcal{A}$ as an abstract linear algebra C such that:

(i) The C-module $\mathcal{W} = \mathcal{X}(C)$ of all C-vectors is the Lorentz C-module.

(ii) There exists a covariant derivative ∇ in $\mathcal{W}$ such that $\nabla g = 0$, where g is the Lorentz scalar product in $\mathcal{W}$.

(iii) $Ric = 0$.

Instead of (iii), let the following condition be satisfied:

(iii′) $Ein + \Lambda g = T$

Here Ein is the Einstein tensor, Λ the cosmological constant, and T a suitable energy-momentum tensor; one now speaks of an *extended Einstein algebra*.

The C-vectors of condition (i) are derivations of the algebra C. The C-module $\mathcal{W}$ of all such C-vectors is said to be a *Lorentz* C-module if a scalar product with the Lorentz signature can be defined in it. This always can be done if $\mathcal{W}$ has a basis $(W_0, W_1, \ldots, W_n)$. Condition (ii) uses the fact that

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for any scalar product g in $\mathcal{W}$ there exists exactly one symmetric derivative ∇ in $\mathcal{W}$ such that $\nabla g = 0$. By using it one can define the corresponding Riemann tensor. Since the existence of a basis $(W_0, W_1, \ldots, W_n)$ is postulated in $\mathcal{W}$, the trace operator is defined, and the Ricci tensor and consequently the Einstein equations can be introduced in $\mathcal{W}$ [condition (iii) or (iii′)]. It can be easily seen that every spacetime manifold satisfying Einstein's equations can be represented as an Einstein algebra, but not *vice versa*. There exist Einstein algebras which are essentially more general than the standard relativistic spacetimes, for instance, spacetimes including some singularities. The goal of the present paper is to study systematically this additional generality offered by the theory of Einstein algebras.

Einstein algebras were originally defined as *abstract* linear algebras, but—without loss of generality—one can constrain oneself to considering only *functional* linear algebras. One can always construct the Gelfand representation of an Einstein algebra $\mathcal{A}$ which is already a functional linear algebra, and is a *universal* representation of $\mathcal{A}$ in the sense that every representation of $\mathcal{A}$ is its subrepresentation [for details see Heller (1992)]. In the following we shall consider only real functional Einstein algebras.

Let M be a nonempty set, and $\mathcal{F}(M)$ a functional algebra defined on M Hausdorff separating points in M (if M does not enjoy this property, we can always introduce a suitable equivalence relation which would induce it). The pair $(M, \mathcal{F}(M))$ is called *Einstein ringed space*, and $\mathcal{F}(M)$ its *structural ring*. Let $\mathcal{A}$ be an Einstein algebra; it can be easily seen that $\gamma: \mathcal{A} \to C^\infty(M)$ is a subrepresentation of the Gelfand representation of $\mathcal{A}$. The pair $(M, C^\infty(M))$ is called a *Geroch ringed space*. It is equivalent to a smooth manifold satisfying Einstein's equations (Geroch, 1972). Thus the theory of Geroch ringed spaces is equivalent to the standard theory of general relativity.

Let $C \subset \mathbf{R}^M$ be the set of all real functions on a nonempty set M such that (i) C is closed with respect to localization, and (ii) C is closed with respect to superposition with smooth functions on $\mathbf{R}^n$ [for definitions of these concepts see Heller (1992)]. The pair (M, C) is called a *differential space* (in the sense of Sikorski), and C a *differential structure* on M. One can correspondingly define the *Sikorski representation* $\sigma: \mathcal{A} \to C$ of an Einstein algebra [one can also speak of a *Sikorski ringed space* (M, C)]. If we additionally assume that M is locally diffeomorphic to $\mathbf{R}^n$, the differential space (M, C) changes into a smooth manifold $(M, C^\infty(M))$, and the corresponding Sikorski ringed space reduces to the Geroch ringed space.

It is obvious that the theory of Sikorski ringed spaces is more general than the standard theory of general relativity. It can be shown (Heller, 1992) that it contains, as its intrinsic elements, all spacetime singularities which do not violate the following property: there exists an open covering $\mathcal{B}$ of M

such that on each open set $B \in \mathcal{B}$ there are n smooth[3] tangent vector fields forming a vector basis. A differential space for which this property holds is said to be of *constant differential dimension.* In particular, all regular singularities[4] can be easily dealt with by using the Sikorski formalism (Heller and Sasin, 1991). Spacetimes with stronger types of singularities cannot be organized into differential spaces admitting a global vector basis (in the above sense). To overcome this difficulty and to include stronger types of singularities into the formalism we should change from Einstein functional algebras to the sheaves of Einstein functional algebras. Such a program was outlined in previous work (Heller, 1992); the present study aims at its systematic development.

In Section 1, we define Einstein structured spaces as sheaves of Einstein algebras satisfying a condition changing them into a workable structure. In Section 2, we discuss those Einstein structured spaces which contain regular and quasiregular singularities, and in Section 3 those which contain curvature singularities. An important theorem is proved giving us an insight into the structure of curvature singularities (considered as the b-boundary of spacetime) over which the fiber of the frame bundle degenerates to a single point. In Section 4, the crucial example of the closed Friedmann universe is discussed.

1. EINSTEIN STRUCTURED SPACES

In the present relativistic paradigm singularities are organized into the so-called singular boundaries. A boundary $\tilde{\partial}M$ of spacetime M is defined to be a set of incomplete curves, the incompleteness itself being treated as a symptom of the existence of singularities. Then an equivalence relation $\rho \subset \tilde{\partial}M \times \tilde{\partial}M$ is introduced dividing the set $\tilde{\partial}M$ into the classes of incomplete curves such that each class defines the single "ideal" point of the singular boundary. In this way, one obtains the singular boundary ∂M of spacetime M, $\partial M = \tilde{\partial}M/\rho$. Various constructions of ∂M differ in the choice of $\tilde{\partial}M$ or in the choice of ρ. To guarantee a suitable "contact" of spacetime M with its singular boundary ∂M a topological condition is assumed stating that M is dense in $\overline{M} := M \cup \partial M$ (Gruszczak and Heller, 1993, Section 3). In this way, various singular boundaries of spacetime have been defined [g-boundary (Geroch, 1968), b-boundary (Schmidt, 1971), p-boundary (Dodson, 1979), essential boundary (Clarke, 1979)]. In the following, we construct a theoretical scheme that incorporates all the above singularity definitions, although some of them in a more natural way than the others.

[3] In the sense of the theory of differential spaces.
[4] In the Ellis and Schmidt (1977) classification of singularities.

Definition 1.1. Let $(M, topM)$ be a topological space. The sheaf $\mathscr{C}$ of real continuous (in $topM$) functions on $(M, topM)$ is a *differential structure* on M if, for any open set $U \in topM$ and any functions $f_1, \ldots, f_n \in \mathscr{C}(U)$, $\omega \in C^\infty(\mathbf{R}^n)$, the superposition $\omega \circ (f_1, \ldots, f_n) \in \mathscr{C}(U)$. The pair $(M, \mathscr{C})$ is called a *structured space* [if necessary, a structured space is also denoted by the triple $(M, topM, \mathscr{C})$].

Definition 1.2. Let $(\overline{M}, top\overline{M})$, where $\overline{M} = M \cup \partial M$, be a topological space such that M is open and dense in $\overline{M}$; ∂M is called the *boundary* of M. *Einstein structured space* is a structured space such that for any $p \in U$, $U \in topM$, where $topM$ is the topology on M induced from that of $\overline{M}$; $\mathscr{C}(U)$ is an Einstein functional linear algebra. Einstein structured spaces are also called *sheaves of Einstein algebras.*

The concept of structured spaces (without naming it) was first introduced by Mostov (1979), and it turned out to be a natural generalization of Sikorski's differential spaces (Heller, 1991; Heller *et al.*, 1992). Any structured space $(M, topM, \mathscr{C})$ naturally becomes a Sikorski differential space if for any $U \in topM$ and any $p \in U$ there exists a bump function, i.e., a function φ such that $\varphi(p) = 1$ and $\varphi | M - U = 0$.

Since $M \in topM$, $\mathscr{C}(M)$ of Definition 1.2 is an Einstein algebra, and the sheaf $\mathscr{C}_M = \mathscr{C} | M$ is locally free, i.e., the $\mathscr{C}(M)$-module $\mathscr{X}(\mathscr{C}_M)$ of cross sections of the sheaf $\mathscr{C}_M$ has a local $\mathscr{C}(M)$-basis. Consequently, $(M, \mathscr{C}_M)$ is a differential space (in the sense of Sikorski) of constant differential dimension. Therefore, when changing from the theory of Einstein algebras to the theory of sheaves of Einstein algebras we gain in generality only by considering the boundary ∂M.

Let $(\overline{M}, \mathscr{C})$ be an Einstein structured space. It can be easily seen that it defines a subrepresentation $\mu: \mathscr{A} \to \mathscr{C} | M$ of the Gelfand representation of an abstract Einstein algebra $\mathscr{A}$.

2. REGULAR AND QUASIREGULAR SINGULAR BOUNDARIES

In this section we deal with regular and quasiregular singularities. As we shall see, the sheaf method will allow us to generalize some standard concepts and, consequently, to make a more detailed classification of singular boundaries.

Definition 2.1. An Einstein structured space is *regular* if $\partial M = \emptyset$ or $M \cup \partial M$ is of constant differential dimension.

In the case of spacetime with regular singularities the corresponding Einstein structured space can be reduced to an Einstein ringed space (Heller, 1992), and all spacetime structures can be naturally extended to $M \cup \partial M$

(Heller and Sasin, 1991). To include quasiregular singularities into our scheme we must introduce some auxiliary concepts.

Let $\mathcal{A}_1 = (M_1, \mathcal{C}_1)$ and $\mathcal{A}_2 = (M_2, \mathcal{C}_2)$ be Einstein ringed spaces. Since M_1 and M_2 are topological spaces, we can introduce the following definition. An *isometry of Einstein ringed spaces* $\mathcal{A}_1$ and $\mathcal{A}_2$ is a homeomorphism f: $M_1 \to M_2$ such that (i) f^*: $\mathcal{C}_2 \to \mathcal{C}_1$ is an isomorphism of algebras, (ii) $g_1(X, Y) = g_2(X_f, Y_f)$, for $X, Y \in \mathcal{X}(M_1)$, where $X_f(\alpha) = X(\alpha \circ f) \circ f^{-1}$, $\alpha \in \mathcal{C}_2$, and g_1 and g_2 are Lorentz metrics in the $\mathcal{C}_1$-module $\mathcal{X}(\mathcal{C}_1)$ and $\mathcal{C}_2$-module $\mathcal{X}(\mathcal{C}_2)$, respectively.

Let $(M_1, \mathcal{C}_1)$ and $(M_2, \mathcal{C}_2)$ be regular Einstein structured spaces. An isometry of sheaves

$$\underline{f}: \quad (M_1, \mathcal{C}_1) \to (M_2, \mathcal{C}_2)$$

is an *isometry of regular Einstein structured spaces* if:

(i) f_V^*: $\mathcal{C}_2(V) \to \mathcal{C}_1(f_U^{-1}(U))$ is an isometry of Einstein ringed spaces, for $V \in topM_2$, $U \in topM_1$.

(ii) $g_{1U}(X, Y) = g_{2V}(X_f, Y_f)$, where $U \in topM_1$, $V = f(U) \in topM_2$, and $X_f(\alpha) = X(f_V^*\alpha) \circ f_V$, $\alpha \in \mathcal{C}_2(V)$.

Now, we are ready to define an *isometry of Einstein structured spaces* as a sheaf isometry

$$\underline{f}: \quad (\overline{M}_1, \mathcal{C}_1) \to (\overline{M}_2, \mathcal{C}_2)$$

such that $\underline{f}|(M_1, \mathcal{C}_1 | M_1)$ is an isomorphism of regular Einstein structured spaces $(M_1, \mathcal{C}_1 | M_1)$ and $(M_2, \mathcal{C}_2 | M_2)$. This definition means that the boundaries ∂M_1 and ∂M_2 are homeomorphic, and M_1 and M_2 are isomorphic.

We say that a rectifiable curve $c[a, b] \to M$ *ends at a point* $p \in \partial M$ if $c(b) = p$ and there exists $\epsilon > 0$ such that $c([b - \epsilon, b)) \subset M$. Let End_p denote the set of all smooth rectifiable curves ending at $p \in \partial M$. We say that a curve $c \in End_p$ is of a *continuous type* if $\lim_{t \to b^-} c(t) = p$.

Definition 2.2. A boundary point $p \in \partial M$ is *quasiregular* if, for any rectifiable curve $c \in End_p$, there exist, $\epsilon > 0$, an open neighborhood $U \in topM$ such that $c([b - \epsilon, b)) \subset U$, and an isomorphic embedding

$$\iota_U: \quad (U, \mathcal{C} | U) \to (N, \mathcal{N})$$

where $(N, \mathcal{N})$ is a regular Einstein structured space of constant differential dimension $dimM = m$.

This definition is more general than the standard quasiregular singularity definition (Ellis and Schmidt, 1977) only in that it allows for a local extension into a regular Einstein structured space instead of a usual smooth manifold.

However, it can be modified in the following way. Let us call an open neighborhood $U \in topM$ *strictly adherent to* $p \in \partial M$ if $p \in clU$.

Definition 2.3. A boundary point $p \in \partial M$ is *adherently quasiregular* if there exists an open neighborhood $U \in topM$ strictly adherent to p such that the Einstein structured space $(clU, \mathscr{C} | clU)$ is a regular Einstein structured space.

It is evident that every adherently quasiregular boundary point is quasiregular, but not *vice versa*. The advantage of adherently quasiregular boundary points is that they are defined with no reference to any curves.

The spacetime of a straight cosmic string can serve as an example of an Einstein structured space with a quasiregular singular boundary. This example was analyzed in Heller (1992, Section 4).

3. CURVATURE SINGULAR BOUNDARIES

Let $(M, \mathscr{C})$ be an Einstein structured space, $p \in \partial M$, and $c \in End_p$. We assume that there exist $\epsilon > 0$, $b \in \mathbf{R}$ such that $c([b - \epsilon, b)) \subset M$ and $c(b) = p$. Let further $E(t) = (e_1, \ldots, e_n)$ be a frame parallelly transported along the curve c, i.e.,

$$E: \quad [b - \epsilon, b) \rightarrow F(M)$$

where $F(M)$ is the frame bundle over M and $t \in [b - \epsilon, b)$.

Definition 3.1. $p \in \partial M$ is said to be a *curvature singularity* if at least one of the components of the Riemann tensor with respect to $E(t)$ diverges.

Let $(\overline{M}, \mathscr{C})$ be an Einstein structured space such that its singular boundary ∂M contains curvature singularities. To answer the question of how the differential structure behaves at such boundaries we must specify a concrete boundary. Since curvature singularities are related to the behavior of moving frames along curves going to such a boundary, our choice is Schmidt's b-boundary of spacetime. First, we show that the b-completion of any spacetime can be organized into an Einstein structured space.

Proposition 3.2. Let (M, g) be a spacetime. On the b-completion $\overline{M}$ of M there exists the differential structure $\overline{\mathscr{C}}$ such that $\overline{\mathscr{C}}(M) = C^\infty(M)$ and if (M, g) is a solution to Einstein's field equations, then $(\overline{M}, \overline{\mathscr{C}})$ is an Einstein structured space.

Proof. By construction. Let (M, g) be a spacetime, $F(M)$ the connected component of the (pseudo-)orthonormal frame bundle over M, and G a Riemann metric on $F(M)$ defined in the standard way by using connection on M (Schmidt, 1971). For a sufficiently large $n \in \mathbf{N}$ there exists an isometric embedding $\iota: F(M) \hookrightarrow \mathbf{R}^n$ such that $G = \iota^*\eta$. Such an embedding is not unique, but all Riemann metrics resulting from all such embeddings are uniformly equivalent (Schmidt, 1971). This fact guarantees that the following boundary construction is independent of the concrete embedding.

The differential structure $C^\infty(F(M))$ on $F(M)$ is generated by the set of projections $\{\pi_1 | F(M), \ldots, \pi_n | F(M)\}$, where π_i: $\mathbf{R}^n \to \mathbf{R}$ are the projections on the ith coordinate.

In the standard way, one constructs the closure $\overline{F(M)} = F(M) \cup \bar{\partial}$, where $\bar{\partial}$ is the Cauchy boundary of $F(M)$. It is evident that $\overline{F(M)}$ is the closure of $F(M)$ in $\mathbf{R}^n$. Therefore, one can define the differential structure $(\mathscr{E}_n)_{\overline{F(M)}}$ on $\overline{F(M)}$ induced from the differential space $(\mathbf{R}^n, \mathscr{E}_n)$ where $\mathscr{E}_n := C^\infty(\mathbf{R}^n)$.

The structural group of the frame bundle, which is the proper Lorentz group $\mathscr{L}_+^\uparrow$, acts on $\overline{F(M)}$. The b-completion of M, in the category of structured spaces, is defined to be the quotient structured space $(\overline{M}, \overline{\mathscr{C}})$, where $\overline{M} = \overline{F(M)}/\mathscr{L}_+^\uparrow$, and $\overline{\mathscr{C}} = (\mathscr{E}_n)_{\overline{F(M)}}/\rho$, ρ being the equivalence relation defined by the action of the group $\mathscr{L}_+^\uparrow$, i.e., $\overline{\mathscr{C}}$ is the sheaf on the topological space $(\overline{F(M)}/\rho, top\overline{F(M)}/\rho)$ given by

$$((\mathscr{E}_n)_{\overline{F(M)}}/\rho)(V) = \{f\colon V \to \mathbf{R}: f \circ \pi | \pi^{-1}(V) \in (\mathscr{E}_n)_{\overline{F(M)}}(\pi^{-1}(V))\}$$

for $V \in top\overline{F(M)}/\rho$, where π is the projection of a point onto its equivalence class (orbit). This ends the proof, but, for the sake of completeness, let us add that the b-boundary of M is defined to be $\partial_b M = \overline{M} - M = \pi(\overline{F(M)}) - \pi(F(M))$. ∎

Theorem 3.3. Let $p_0 \in \partial_b M$. If the fiber $\pi^{-1}(p_0)$ degenerates to a single point, i.e., if the entire boundary $\overline{F(M)} - F(M)$, $F(M) \subset \mathbf{R}^n$, is a single orbit over p_0 of the group action $\mathscr{L}_+^\uparrow \times \overline{F(M)} \to \overline{F(M)}$, then the only global cross sections of Einstein structured space $(\overline{M}, \overline{\mathscr{C}})$ are constant functions, i.e., $\overline{\mathscr{C}}(\overline{M}) \cong \mathbf{R}$, and the only open neighborhood of p_0 is the entire $\overline{M}$.

Proof. Let $\Gamma(\overline{M})$ be the set of all global cross sections of an Einstein structured space $(\overline{M}, \overline{\mathscr{C}})$, and let us consider $\alpha \in \Gamma(\overline{M}) = C^\infty(\overline{F(M)}/\mathscr{L}_+^\uparrow)$. Then $\tilde{\alpha} = \alpha \circ \pi \in C^\infty(\overline{F(M)})$ is constant on orbits.

Let $\{e_n\} \in \pi^{-1}(p)$, $p \in M$, be the set of frames convergent to $e \in \overline{F(M)}$ (such a point always exists by the definition of closure). Of course, one has $\tilde{\alpha}(e_n) = c_1$ for any $n \in \mathbf{N}$.

From continuity of $\tilde{\alpha}$ it follows that

$$\lim_{n\to\infty} \tilde{\alpha}(e_n) = \tilde{\alpha}(e) = \alpha(p_0)$$

The last equality follows from the fact that $e \in \pi^{-1}(p_0)$. On the other hand,

$$\lim_{n\to\infty} \tilde{\alpha}(e_n) = c_1$$

since it is a constant sequence. Therefore, $\alpha(p_0) = c_1$.

Let us notice that

$$\tilde{\alpha}(e_n) = \alpha \circ \pi(e_n) = \alpha(p)$$

Hence, $\alpha(p) = \alpha(p_0)$.

To prove the second part of the theorem let us assume that V is an open neighborhood of the point p_0 (in the quotient topology). $\pi^{-1}(V)$ is an $\mathscr{L}_+^\uparrow$-invariant open set in $\overline{F(M)}$ [i.e., if $\pi^{-1}(V)$ contains a point, it also contains its orbit]. It follows that $\pi^{-1}(V)$ contains the boundary $\overline{F(M)} - F(M)$, and it is of the form $W \cap \overline{F(M)}$, where $W \in top\mathbf{R}^n$, $(\overline{F(M)} - F(M) \subset W$.

There exists a cross section s: $M \to F(M)$ such that $s(M) \subset \pi^{-1}(V)$. Since the set $\pi^{-1}(V)$ is $\mathscr{L}_+^\uparrow$-invariant and contains values of a certain cross section, it must coincide with the entire space $\overline{F(M)}$. This is equivalent to the fact that $V = \overline{M}$. ∎

Corollary 3.4. Let the condition of Theorem 3.3 be satisfied. The b-boundary $\partial_b M$ of M consists of a single point.

Proof. The differential structure $\overline{\mathscr{C}}$ on $\overline{M} = M \cup \partial_b M$, consisting only of constant functions, does not distinguish points. ∎

Properties of the b-boundary $\partial_b M$, expressed in Theorem 3.3 and Corollary 3.4, are not disastrous for spacetime M itself. Since M is open (and dense) in $\overline{M}$, the sheaf $\overline{\mathscr{C}}(M)$ contains enough functions to define the smooth manifold structure on M. However, from all these functions only constant functions can be prolonged to $\overline{M}$. This fact shows a disastrous character of singularities. In Heller *et al.* (1992) a singular boundary point p_0 such that the fiber $\pi^{-1}(p_0)$ degenerates to a single point was called a *malicious boundary point* (or a *malicious singularity*).

In view of the above, it would be useful to have an independent criterion telling us when a singular fiber degenerates to a single point. To find such a criterion one can turn to the holonomy group. Clarke (1978) defined the *singular holonomy group* G_p at p, $p \in \partial_b M$, which essentially is the group of Lorentz transformations that are generated by the parallel transport around arbitrary short loops near p. Any $\pi^{-1}(p)$, for $p \in \partial_b M$, is a homogeneous space of the form $\mathscr{L}_+^\uparrow/G_p$. If $G_p = \mathscr{L}_+^\uparrow$, the fiber $\pi^{-1}(p)$ is reduced to a single point. This leads to the following result.

Corollary 3.5. If $G_p = \mathscr{L}_+^\uparrow$, then $p \in \partial_b M$ is the malicious boundary point. ∎

4. EXAMPLES

A. Two-Dimensional Closed Friedmann Universe (Bosshard, 1976; see also Dodson, 1978, Chapter III,3). Let us consider a spacetime

$$N = \{(\eta, \chi) \colon \eta \in (0, T), \chi \in S^1\}$$

where $(0, T) \subset \mathbf{R}$, with the metric

$$ds^2 = R^2(\eta)(-d\eta^2 + d\chi^2)$$

We assume that $R^2(\eta) \to 0$ as $\eta \to 0$. Without loss of generality we assume also that $R(\eta) \sim \eta^\rho$, for some $\rho > 0$, as $\eta \to 0$. This behavior corresponds to the initial singularity.

The positively oriented component of the orthonormal frame bundle O^+N over N is of the form

$$O^+N = \left\{ \left(\eta, \chi, \begin{bmatrix} \cosh \lambda & \sinh \lambda \\ \sinh \lambda & \cosh \lambda \end{bmatrix}, \frac{1}{R} \begin{bmatrix} \partial_\eta \\ \partial_\chi \end{bmatrix} \colon (\eta, \chi) \in N, \lambda \in \mathbf{R} \right\}$$

and the Riemann metric on O^+N [constructed as in Schmidt (1971)] is

$$ds^2 = R^2(\eta)(d\eta^2 + d\chi^2) \cosh 2\lambda - 2\, d\eta\, d\chi \sinh 2\lambda + \left(\frac{\dot{R}(\eta)}{R(\eta)}\, d\chi + d\lambda \right)^2$$

By using the construction given in the proof of Proposition 3.2 one could obtain the structured space $(\overline{N}, \overline{\mathscr{C}})$, $\overline{N} = N \cup \partial_b N$, representing spacetime of the two-dimensional closed Friedmann model with its b-boundary. However, we shall use Theorem 3.4 to show that $\overline{\mathscr{C}}$ consists only of constant functions.

Let us consider the initial singularity at $\eta = 0$. We fix χ_0 in S^1 and construct the sequence $s \colon N \to O^+N$

$$n \mapsto (\eta_n, \chi_0, 0)$$

for some $\eta_n \in (0, T)$ such that $\eta_n \to 0$. For all $n > n_0 \in \mathbf{N}$ this sequence lies on the horizontal curve $c \colon [0, 1) \to O^+N$

$$t \mapsto (1 - t, \chi_0, 0)$$

Indeed, for all $n > n_0$ there exists $t_n \in [0, 1)$ such that $\eta_n = 1 - t_n$ and $t_n \to 1$.

We have

$$\|\dot{c}(t)\| = R(1 - t)$$

For large enough $k, n \in \mathbf{N}$ we obtain

$$d(s(n), s(k)) \le \left| \int_{t_n}^{t_k} \|\dot{c}(t)\|\, dt \right| \sim \left| \int_{\eta_n}^{\eta_k} \eta^\rho\, d\eta \right|$$

By using this inequality it can be easily checked that s is a Cauchy sequence.

Its equivalence class determines a point $\tilde{p}_0$ of the Cauchy boundary of O^+N. Consequently, $p_0 = \pi(\tilde{p}_0) \in \partial_b N$ is the initial singularity at $\eta = 0$, $\chi = \chi_0$.

Similarly, the sequence r: $\mathbf{N} \to O^+N$ given by

$$n \mapsto (\eta_n, \chi_0 + \delta_n, \lambda_1)$$

where $\eta_n \to 0$, $\delta_n \to 0$, and χ_0, λ_1 are fixed, is another Cauchy sequence determining a point $\tilde{p}_1 \in \overline{O^+N}$. One has $\pi(\tilde{p}_1) = \pi(\tilde{p}_0) = p_0 \in \partial_b N$. Therefore, $\tilde{p}_1$ and $\tilde{p}_0$ belong to the same fiber.

The horizontal curve k_n: $[0, \delta_n] \to O^+N$ given by

$$t \mapsto \left(\eta_n, \chi_0 + t, -\frac{\dot{R}(\eta_n)}{r(\eta_n)}\, t\right)$$

joins $s(n)$ and $r(n)$ for every $n \in \mathbf{N}$, and its length is

$$K_n = \left| R_n \int_0^{\delta_n} \left[\cosh\left(-\frac{2\dot{R}(\eta_n)}{R(\eta_n)}\, t\right)\right]^{1/2} dt \right|$$

$$= \left| \frac{R^2(\eta_n)}{\dot{R}(\eta_n)} \int_0^{\lambda_1} (\cosh 2\lambda)^{1/2}\, d\lambda \right|$$

$$\leq \left| \sqrt{2}\, \frac{R^2(\eta_n)}{\dot{R}(\eta_n)} \sinh \lambda_1 \right|$$

The last inequality follows from

$$0 < \cosh \lambda \leq (\cosh 2\lambda)^{1/2} \leq \sqrt{2} \cosh \lambda$$

Taking into account that $\eta \to 0$ as $\eta \to 0$, we obtain

$$K_n \to |\sqrt{2} \sinh \lambda_1 \eta_n^{\rho+1}|$$

Since $\rho > 0$, $K_n \to 0$ as $n \to \infty$.

Therefore, we have established that, in the limit as $n \to \infty$, $d(\tilde{p}_0, \tilde{p}_1) = 0$, and consequently $\tilde{p}_0 = \tilde{p}_1$. However, these points were chosen arbitrarily. Hence, the fiber $\pi^{-1}(p_0)$ degenerates to a single point.

If we additionally assume that $R(\eta) \to 0$ as $\eta \to T$, $T \in \mathbf{R}^+$, we have the final singularity in the model. By Corollary 3.4 this singularity together with the initial singularity at $\eta = 0$ constitute the same point of the b-boundary. From the viewpoint of the theory of structured spaces this should be interpreted in the following way. From the fact that $\mathscr{C}(\overline{M}) \cong \mathbf{R}$ it follows that the set $\mathscr{X}(\overline{M})$ of all vector fields on $\overline{M}$ consists of zero vector fields (i.e., only zero vector fields can be prolonged from M to $\overline{M}$), and consequently the "bundle length" of any curve joining the initial and final singularities is zero. This shows an "inaccessible" character of these singularities. But from the point of view of any observer who does not "touch" the boundary, there are no pathologies at all (see also Heller and Sasin, 1993, 1994).

B. Two-Dimensional Schwarzschild Spacetime (Johnson, 1977). A similar analysis can be carried out for the two-dimensional Schwarzschild spacetime with the metric

$$ds^2 = -\frac{dr^2}{(2/r - 1)^{1/2}} + r\,d\chi^2$$

In this model, the singularity is at $r = 0$. The fiber over it degenerates to a single point, and Theorem 3.3 applies.

C. Four-Dimensional Closed Friedmann Universe and Four-Dimensional Schwarzschild Spacetime. The above two examples can be generalized to the corresponding four-dimensional cases. In fact, in original papers studying these cases (Bosshard, 1976; Johnson, 1977) the "reduction" of four-dimensional Friedmann and Schwarzschild spacetimes to two dimensions was used to demonstrate the degeneracy of the fibers over singularities in these models (see also Dodson, 1978, Chapter III,3).

5. CONCLUDING REMARKS

In the previous work (Heller, 1992) and in the present one we have proposed an algebraic generalization of Einstein's general relativity. Our theory works on more general spaces than traditional differentiable manifolds. If one limits oneself to domains without singularities, the standard theory of relativity is recovered. To deal with milder types of singularities (such as regular singularities) one functional algebra, called Einstein algebra, is enough (this case was considered in the previous paper); to cover all types of singularities one must change to a sheaf of Einstein algebras. As examples of the closed Friedmann world model and the Schwarzschild spacetime show, Schmidt's b-boundary construction turns out to be useful. "Pathologies" discovered in these cases by Bosshard (1976) and Johnson (1977) become a useful tool of analysis when considered in the category of sheaves of linear algebras (see above, Section 4).

One could expect that some sorts of singularities can also appear when one changes to the quantum gravity regime: the smooth manifold structure of spacetime can break down and some more general spaces can enter the scene. So far our algebraic approach in this area has been used in two ways. First, the idea that in quantum gravity smooth manifolds could be replaced by Einstein algebras was explored by Heller (1993) and by Kull and Treumann

(1995). We should notice that in such a case the Einstein equations are automatically satisfied. Second, one can also change from commutative to noncommutative Einstein algebras, and try to develop a formalism analogous to that used in quantum field theory. This approach was hinted at by Heller (1993), and the working model was constructed by Parfionov and Zapatrin (1994).

An important task would be to find experimental predictions of our generalized theory. They should be looked for in quantum gravity domains in which the manifold structure of spacetime breaks down. There is theoretical evidence (Heller, 1993) that they could be connected with the fact that in such domains the equivalence principle in its usual formulation is not valid. One should expect that it will be replaced by some generalization. The theory of Einstein algebras opens some possibilities.

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